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THE ISOPERIMETRIC PROFILE OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

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Abstract

We compute, up to a multiplicative constant, the isoperimeric profile of (non-compact) homogeneous Riemannian manifolds by constructing "explicit" exhaustions which give estimates for the distribution of the volume. For those Riemannian manifolds only three very different isoperimetric profiles exist and the isoperimetric profile governs the asymptotic of the heat kernel decay on the diagonal and vice-versa. By discretisation, the isoperimetric profiles of finitely generated discrete subgroups of Lie groups are also computed.

1. Introduction

1.1 The classical isoperimetric profile

Let X be a complete Riemannian manifold and let $0 < t < vol(X)$. The isoperimetric profile of X is

$$I_X(t) = \inf_{vol(\Omega)=t} vol(\partial\Omega),$$

where the infimum is taken over relatively compact domains Ω with regular boundary $\partial\Omega$. Compare [5, pp. 140-143], [16, 4.74]. With the exception of the simply connected constant sectional curvature spaces $\mathbb{R}^n, S^n, H^n$ where the infimum is realised by balls, the exact computation of $I_X(t)$ even for familiar Riemannian manifolds may be problematic. For example, in the case of real projective spaces with the locally spherical metric, the answer is known and proved only in dimension 2 and

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3 [5, p.141]. See [36], [35] for estimates in the case of periodic metrics covering the torus.

1.2 Asymptotic invariants

At the beginning of the 80s, M. Gromov [20], [17] initiated the study of non-bounded Riemannian manifolds and other metric spaces including infinite finitely generated groups with word metrics, up to quasi-isometry. The emphasis is then on the asymptotic invariants which capture the large scale geometry of the space [22] and not on the local structure of the space. For example, the universal cover of a compact Riemannian manifold and its fundamental group with a word metric are essentially viewed as the same object. On a finitely generated group Γ with finite symmetric generating set S, the discrete version of the isoperimetric profile is

$$I_\Gamma(n) = \min_{|\Omega|=n} |\partial\Omega|,$$

where the minimum is taken over subsets Ω of Γ of cardinality n and where

$$\partial\Omega = \{\gamma \in \Omega : \exists s \in S : \gamma s \in \Gamma \setminus \Omega\}.$$

See [22] 0.5 and 5.E. From this unified viewpoint, the isoperimetric profile becomes a computable asymptotic invariant -at least for homogeneous Riemannian manifolds and finitely generated discrete subgroups of Lie groups; see Theorem 2.1, which is strongly related to the heat kernel decay. On a finitely generated group Γ with finite symmetric generating set S, the discrete heat kernel is given by the probabilities

$$p_n(x,y) = \sum_{z_0,\dots,z_n} p(z_0,z_1)\dots p(z_{n-1},z_n),$$

where the sum is taken over the $n+1$-uples of Γ with $z_0 = x$ and $z_n = y$, and where $p(x,y) = |S|^{-1}$ if $x^{-1}y \in S$ and $p(x,y) = 0$ if this is not the case.

1.3 Amenability according to Følner and Kesten

In 1955 Følner [14] proved that a finitely generated group Γ is non-amenable if and only if for any finite symmetric generating set S there exists an $\epsilon > 0$ such that

$$|\partial\Omega| \geq \epsilon|\Omega|$$

for all finite subsets Ω in Γ. In 1959 Kesten [29] proved that a finitely generated group Γ is non-amenable if and only if for any finite symmetric generating set S of Γ, there is a constant $C > 0$ such that

$$p_n(e,e) \leq C \exp(-\frac{n}{C})$$

for all $n \in \mathbb{N}$. In 1968 Milnor [32] pointed out relations among the curvature of a compact Riemannian manifold, the growth of its fundamental group and the decay of $p_n(e,e)$. The problem he raised at the end of the paper, namely whether $p_n(e,e)$ has an exponential decay if Γ is the fundamental group of a compact negatively curved Riemannian manifold, has a natural solution relying on the isoperimetric profile. Indeed, let x_0 be a base point in the universal cover $\tilde{\mathrm{X}}$ of X and consider the unit vector field

$$Z(x) = grad(d(x, x_0))$$

defined on $\tilde{\mathrm{X}} \setminus \{x_0\}$ where $d(x, x_0)$ is the distance between x and x_0. As the curvature of X is negative, it follows that there is an $\epsilon > 0$ such that for all $x \in \tilde{\mathrm{X}} \setminus \{x_0\}$ we have

$$div Z(x) > \epsilon.$$

Let ω be the volume form on $\tilde{\mathrm{X}}$. If Ω is a domain in $\tilde{\mathrm{X}}$ with regular boundary $\partial\Omega$, then the divergence version of Stokes formula shows that

$$\epsilon \times vol(\Omega) \leq \int_\Omega div Z\omega = \int_{\partial\Omega} i_Z\omega \leq vol(\partial\Omega).$$

This implies the corresponding inequality

$$|\partial\Omega| \geq \epsilon|\Omega|$$

(with another constant ϵ) for subsets Ω of the fundamental group of X . To prove it, consider the orbit of a ball of radius $r = diam(X)$ in $\tilde{\mathrm{X}}$ under the action of the subset $\Omega \subset \Gamma$ and the generating set of Γ consisting of the elements $\gamma \in \Gamma$ such that $d(\gamma x_0, x_0) \leq 2r + 1$. The results of Følner and Kesten mentioned above imply the exponential decay of $p_n(e,e)$. Compare with [3], [22, 0.5 C], [38]. Another proof reads as follows. The fundamental group of a compact negatively curved manifold contains a free subgroup on two generators [21, 5.3 E], [15, 1.32, 1.38] and the probability of returning to the origin cannot decrease when passing to a (finitely generated) subgroup [39].

1.4 Varopoulos links between growth, isoperimetry and heat decay

In 1982 Pansu [34] established the lower bound

$$I(t) \geq \frac{t^{\frac{3}{4}}}{C}$$

for $t \geq 1$ and C a constant for the isoperimetric profile of the 3-dimensional Heisenberg group with a left-invariant Riemannian metric and the corresponding lower bound for the lattices of this Lie group. Around the middle of the 80s Varopoulos [51] computed the asymptotics of the heat kernel diffusion on Lie groups with polynomial growth. He showed that the following conditions are equivalent on a connected Lie group with a left-invariant Riemannian metric:

$$\exists A \geq 1 : \forall t \geq 1, \frac{t^d}{A} \leq vol(B_t) \leq At^d,$$

$$\exists B \geq 1 : \forall t \geq 1, \frac{t^{-\frac{d}{2}}}{B} \leq p_t(e,e) \leq Bt^{-\frac{d}{2}},$$

where d is an integer and B_t is the Riemannian ball of radius t. He also discovered close relations between the behaviour of the heat kernel diffusion and Sobolev inequalities. A geometric application of this work is that on a Lie group with polynomial growth of degree d, the isoperimetric profile (with respect to a left-invariant Riemannian metric) satisfies

$$I(t) \geq \frac{t^{\frac{d-1}{d}}}{C}$$

for $t \geq 1$ and a constant C. See [51] 0.6, p. 348 and references.

1.5 The inequality of Coulhon and Saloff-Coste

In [11] Coulhon and Saloff-Coste give a direct proof, i.e., not using the heat kernel decay- of this inequality. Moreover, their argument gives a lower bound for the isoperimetric profile starting from an arbitrary growth of the balls. For example, if the growth is exponential, the lower bound they get is

$$I(t) \geq \frac{t}{C\log(t)},$$

where $t \geq 1$ and C is a constant. In [37] and [41] it is shown that the above inequality is essentially optimal in the case of polycyclic groups with exponential growth. This raises the question of the existence of finitely generated amenable groups with an isoperimetric profile bigger than

$$\frac{t}{\log(t)}$$

and shows also that to find such an example, the information on the growth is of no help. In [53], [54], [56], the asymptotic of the heat kernel decay for a general Lie group is computed. See also [24] and [2]. A surprising $\frac{1}{3}$ exponent appears:

$$\frac{\exp(-Ct^{\frac{1}{3}})}{C} \leq p_t(e,e) \leq C\exp(-\frac{t^{\frac{1}{3}}}{C}),$$

where $t \geq 1$ and $C \geq 1$ is a constant. Among connected Lie groups, these asymptotic characterises the unimodular amenable groups with exponential growth (see Theorem 2.1).

1.6 The functional equation of Coulhon and Grigor'yan

In [18] Grigor'yan and [7] Coulhon obtain, in an abstract setting, a refined link between the asymptotic of the heat kernel diffusion and the first eigenvalue of compact domains for the Dirichlet problem. See also [9]. Their result implies that, under suitable hypotheses on the space X, the isoperimetric profile $I(t)$ is given (up to multiplicative constants and for $t \geq 1$) by

$$t\sqrt{\Lambda(t)},$$

where $\Lambda(t)$ is defined by the functional equation

$$\frac{d}{dt}\log(p_t^{-1}) = \Lambda(p_t^{-1}),$$

where $p_t = p_t(x,x)$ is the heat kernel and $x \in X$ is a base point. The present paper can be regarded as a first step to find out which spaces fulfil the above "suitable" hypotheses. On a homogeneous Riemannian manifold, the above relation between the isoperimetric profile and the heat kernel is true. We invite the reader to check the predicted correspondences in the three cases of Theorem 2.1. The lambda functions are respectively $\Lambda(x) = C$ where $C > 0$ is a constant, $\Lambda(x) = x^{-\frac{2}{d}}$ where d is a positive integer and $\Lambda(x) = \log(x)^{-2}$. To summarize, the first case corresponds (in the discrete setting of finitely generated groups) to the equivalence obtained by applying the theorems of Følner and Kesten cited above. The second case corresponds (for Lie groups) to the equivalences first established through Sobolev inequalities by Varopoulos as mentioned above. The third equivalence gives a geometric explanation for the exponent $\frac{1}{3}$ in the decay

$$\exp(-t^{\frac{1}{3}})$$

if we succeed in making clear what a $\frac{t}{\log(t)}$ isoperimetric profile means. We also prove that the discrete analogous correspondence is true for finitely generated discrete subgroups of Lie groups.

1.7 The homogeneity assumption

With no homogeneity assumption it is not true that the decay of the heat kernel determines the isoperimetric profile. For example, in [49] it is shown, under geometric finiteness assumptions that if

$$I(t) \geq \frac{t^{\frac{d-1}{d}}}{C}$$

for $t \geq 1$ and for a constant C, then

$$\sup_{x,y} p_t(x,y) \leq Ct^{-\frac{d}{2}}$$

(for another constant C) but the best known converse seems to be that if

$$\sup_{x,y} p_t(x,y) \leq Ct^{-\frac{d}{2}}$$

then

$$I(t) \geq \frac{t^{\frac{\frac{d}{2}-1}{\frac{d}{2}}}}{C}$$

(again for another constant C). See [52]. In [6], Carron constructs a complete Riemannian manifold with positive injectivity radius and bounded sectional curvature with

$$\sup_{x,y} p_t(x,y) \leq t^{-\frac{d}{2}},$$

and he shows that the inequality

$$I(t) \geq \frac{t^{\frac{\alpha-1}{\alpha}}}{C}$$

for C a constant and $t \geq 1$ *fails* for any $\alpha > \frac{d}{2}$. See also [10].

1.8 Questions and speculations

We conclude this introduction with a general question. Let X be a connected complete Riemannian manifold such that the isometry group G of X acts quasi-transitively, that is, such that the orbit space $G\backslash X$ is compact. (Here we do not assume that G is connected.) Let $x \in X$ be a base point. Let $p_t = p_t(x,x)$ be the heat kernel of X on the diagonal at x. Is the isoperimetric profile of X given (up to multiplicative constants and for large t) by

$$t\sqrt{\Lambda(t)}$$

where $\Lambda(t)$ is defined by

$$\frac{d}{dt}\log(p_t^{-1}) = \Lambda(p_t^{-1})?$$

Notice that the case of quasi-homogeneous Riemannian manifolds (as defined above) contains (after suitable discretisations, see [39]) the case of finitely generated groups. Let us recall why. If F is a finitely generated free group and if $\Gamma = F/R$ is our finitely generated group, let B be a compact Riemannian manifold with $\pi_1(B) = F$. The Galois group Γ of the covering $X \to B$ corresponding to the normal subgroup $R \triangleleft F$ is quasi-isometric to X. Hence the asymptotics of the isoperimetric profile and of the heat kernel on X are essentially the same as the asymptotics of their discrete analogs on Γ. Already for metabelian groups, the asymptotics of $p_n(e,e)$ and $I(n)$ can be quite complicated. It turns out that there are many solvable groups with an isoperimetric profile much bigger than $\frac{n}{\log(n)}$. See [42]. Finding the sharp lower bound for their isoperimetric profile seems to be a difficult task. For example the metabelian wreath products $F \wr Z^k$ where F is a finite abelian group have

$$p_n(e,e) \sim \exp(-n^{\frac{k}{k+2}}),$$

where $\sim$ means bounded above and below up to multiplicative constants and up to periodicity problems. See [42]. Hence $\Lambda(x) = \log(x)^{-\frac{2}{k}}$ so that we expect

$$I(n) \sim \frac{n}{\log(n)^{\frac{1}{k}}}.$$

What has been proved is that there exists a constant $C \geq 1$ such that

$$I(n) \geq \frac{n}{C\log(n)^{\frac{2}{k}}}$$

for all $n \in \mathbb{N}$ and that for many $n \in \mathbb{N}$

$$I(n) \leq C \frac{n}{\log(n)^{\frac{1}{k}}}.$$

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2. Statement of the results

2.1 Three classes of homogeneous Riemannian manifolds

We will use the following notation. If $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ are functions, we write

$$f(t) \preceq g(t),$$

if there exist constants $A \geq 1$ and $B > 0$ such that for all $t \geq 1$ we have

$$f(t) \leq Ag(Bt).$$

If $f(t) \preceq g(t)$ and $g(t) \preceq f(t)$ we write $f(t) \sim g(t)$ and we say that f and g have the same asymptotic behavior.

Theorem 2.1. *Let X be a connected non-compact Riemannian manifold which is homogeneous in the sense that its group of isometries acts transitively on it. Then three cases occur and they can be distinguished either geometrically by the isoperimetric profile, or analytically by the large time decay of the heat kernel on the diagonal, or algebraically by the structure of the identity component G of the group of isometries of X as follows:*

1. $I(t) \sim t \Leftrightarrow p_t \sim \exp(-t) \Leftrightarrow G$ *is non-amenable or non-unimodular.*

2. $I(t) \sim \frac{t}{\log(t)} \Leftrightarrow p_t \sim \exp(-t^{\frac{1}{3}}) \Leftrightarrow G$ *is amenable unimodular with exponential growth.*

3. $I(t) \sim t^{\frac{d-1}{d}}, d \in \mathbb{N} \Leftrightarrow p_t \sim t^{-\frac{d}{2}}, d \in \mathbb{N} \Leftrightarrow G$ *has polynomial growth of degree d.*

In terms of the structure of G, cases 2) and 3) above have more algebraic descriptions but they are also more complicated to state. Up to taking quotients by compact subgroups and up to taking co-compact subgroups (see Proposition 3.6) we are left with a simply connected solvable Lie group S. To such a group correspond naturally a simply connected nilpotent group N and an abelian group T. The group T acts both on S and on N with the property that the corresponding semi-direct products are equal: $ST = NT$. If T is non-compact, we are in case 2). If T is compact, we are in case 3) and the integer d is given by

$$d = \sum_{i=1}^{c} i \times rank(N^i/N^{i+1}),$$

where N^i is the *ith* term in the descending central series of N. See [1]. (There is a missing hypothesis in Th. 4.2 in [1]. Nevertheless, as T acts via the adjoint action by semi-simple transformations, the semi-direct product NT has exponential growth if and only if T is non-compact.)

2.2 Exhaustions by Følner sets and distribution of the volume

Much of the paper is devoted to prove the upper bounds in 2) and 3) for the isoperimetric profile. We prove the following.

Proposition 2.1. *Let X be a connected homogeneous Riemannian manifold. There exists a family of compact submanifolds Ω_t of maximal dimension (with corners) which exhaust the space,*

$$X = \bigcup_{t\geq 1} \Omega_t$$

and $\Omega_s \subset \Omega_t$ if $s \leq t$. Moreover, each set Ω_t is itself exhausted by a family of compact submanifolds $\Lambda_{t,\epsilon}$ of maximal dimension (with corners) with the following properties:

$$\Omega_t = \bigcup_{0\leq\epsilon\leq t} \Lambda_{t,\epsilon}$$

and $\Lambda_{t,\epsilon'} \subset \Lambda_{t,\epsilon}$ if $\epsilon' \geq \epsilon$, $\Lambda_{t,0} = \Omega_t$ and there is a constant $c > 0$ (depending on X only) such that

$$d(\partial\Lambda_{t,\epsilon'}, \partial\Lambda_{t,\epsilon}) \geq c(\epsilon' - \epsilon).$$

If the connected component of the identity G of the group of isometries of X is unimodular amenable with exponential growth, then there is a simply connected unimodular solvable Lie group S which is quasi-isometric to X and for all $t \geq 1$

$$vol(\Lambda_{t,\epsilon}) = c(\lambda^t - \epsilon\mu^t)^d(t-\epsilon)^r,$$

$$vol(\partial\Omega_t) \leq C(\lambda^t)^d t^{r-1},$$

where $c > 0$, $C > 1$, $\lambda > \mu > 1$ are constants d is the growth degree of the simply connected nilpotent commutator group $[S,S]$, and r is the rank of S in the sense that $S/[S,S] \simeq \mathbb{R}^r$. If G has polynomial growth of degree d, then for all $t \geq 1$

$$vol(\Lambda_{t,\epsilon}) = c(t-\epsilon)^d,$$

$$vol(\partial\Omega_t) \leq Ct^{d-1},$$

where $c > 0$, $C > 1$ are constants.

Remark 2.1. In the case of polynomial growth the sets $\Lambda_{t,\epsilon}$ are related to the Ω_t by

$$\Lambda_{t,\epsilon} = \Omega_{t-\epsilon D}$$

for some $D > 0$. In the case of exponential growth there is no $s = s(t,\epsilon)$ such that the sets $\Lambda_{t,\epsilon}$ which exhaust $\Lambda_{t,0}$ are equal to $\Lambda_{s,0}$.

Remark 2.2. In the case of a unimodular amenable Lie group G with exponential growth, the integers $d \geq 1$ and $r \geq 1$ are *not* naturally associated to the bilipschitz equivalence class of X. The simplest way to see this is perhaps to consider the universal cover of the connected component of the identity of the group of rigid motions of the plane. Any left-invariant Riemannian metric on it is bilipschitz equivalent to the three-dimensional Euclidean space (in fact both groups are regular covers of the 3-torus) but its commutator subgroup is isomorphic to $\mathbb{R}^2$. (To obtain an example with exponential growth we can take the Cartesian product of this group with a unimodular solvable group with exponential growth.) Nevertheless, the existence only of $d \geq 1$ and $r \geq 1$ is enough to imply the optimal lower bound

$$p_t(e,e) \geq \frac{\exp(-Ct^{1/3})}{C}.$$

See [8].

2.3 Balls are not Følner sets

Notice that in the case where G is unimodular amenable with exponential growth, an exhaustion of the space by a family of balls gives no upper bound for the isoperimetric profile. The reason for this is easily explained in the discrete setting [30, 5.4. p.479]. See also [13, Ch.VII]. First recall the following from calculus.

Lemma 2.1. *Let a_n be a sequence of positive numbers. If*

$$\lim_{n\to\infty} a_{n+1}/a_n = a,$$

then

$$\lim_{n\to\infty} a_n^{1/n} = a.$$

Lemma 2.2. *Let Γ be a group generated by a finite symmetric set S. Let B_n be the ball of radius n in Γ for the word metric associated with S. The growth of Γ is exponential if and only if there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$*

$$|\partial B_n|/|B_n| > \epsilon.$$

Proof. It is more convenient here to work with the external boundary of subsets of Γ. That is if $\Omega \subset \Gamma$ we define

$$\partial_1\Omega = V_1(\Omega) \setminus \Omega,$$

where

$$V_1(\Omega) = \{\gamma \in \Gamma : d_S(\gamma, \Omega) \leq 1\}.$$

Notice that

$$|\partial_1\Omega|/|S| \leq |\partial\Omega| \leq |S||\partial_1\Omega|.$$

As $V_1(B_n) = B_{n+1}$ we have

$$|B_{n+1}|/|B_n| = 1 + |\partial_1 B_n|/|B_n|.$$

Hence, if $|\partial_1 B_n|/|B_n| > \epsilon$, then $|B_{n+1}| > (1+\epsilon)|B_n|$ so that the growth is exponential. Suppose there exists $\lambda > 1$, such that for all $n \in \mathbb{N}$, $|B_n| > \lambda^n$. Let $b_n = |B_n|$. As $b_n^{1/n} > \lambda$ for all $n \in \mathbb{N}$, the following statement is true. For any (small) $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$, such that if $n \geq N_\epsilon$ then $b_{n+1}/b_n > \lambda - \epsilon$. Hence if $n \geq N_\epsilon$

$$|\partial_1 B_n|/|B_n| > \lambda - \epsilon - 1.$$

q.e.d.

The estimate

$$vol(\partial\Omega_t) \le Ct^{\frac{d-1}{d}}$$

in case 3) of Theorem 2.1 above with Ω_t a ball of radius t, although probably true, has not been proved (to the best of our knowledge) for a general simply connected nilpotent Lie group. Compare [46], [41]. The fact that $I(t) \sim t^{\frac{d-1}{d}}$ for a simply connected nilpotent Lie group of growth d is stated in [22] 5EB.

2.4 Discretisations

As explained in the introduction, on a finitely generated group with a chosen finite symmetric generating set, the discrete isoperimetric profile and the discrete heat kernel are easily defined. But the equivalence relations which turn them into quasi-isometric invariants are somewhat technical. In any cases, the explicit analog of Theorem 2.1 we are able to prove for finitely generated discrete subgroups of Lie groups is the following. (Proposition 2.1 also has a discrete version.) See Theorem 3.5 in [58] for a statement in the case of quasi-transitive graphs.

Corollary 2.1. *Let Γ be a finitely generated group which is a discrete subgroup of a Lie group having a finite number of connected components. Then the following three cases occur. (We fix a finite symmetric generating set S and C is a constant which depends only on Γ and S. The following inequalities are true for all $n \in \mathbb{N}$.)*

1. $|\partial\Omega| \ge \frac{|\Omega|}{C}, \forall\Omega \subset \Gamma \Leftrightarrow p_n(e,e) \le C\exp(-\frac{n}{C}) \Leftrightarrow \Gamma$ *is non-amenable.*

2. $$|\partial\Omega| \ge \frac{|\Omega|}{C\log|\Omega|}, \forall\Omega \subset \Gamma$$

   and there is a family Ω_n such that

   $$\bigcup_{n\in\mathbb{N}} \Omega_n = \Gamma$$

   with $\Omega_m \subset \Omega_n$ if $m \le n$ and

   $$\frac{n}{C} \le |\Omega_n| \le Cn,$$

   $$\frac{n}{C\log(n)} \le |\partial\Omega_n| \le \frac{Cn}{\log(n)},$$

$\Leftrightarrow p_n(e,e) \leq C\exp(-\frac{n^{\frac{1}{3}}}{C})$ *and* $p_{2n}(e,e) \geq \frac{\exp(-Cn^{\frac{1}{3}})}{C} \Leftrightarrow \Gamma$ *contains a finite index polycyclic subgroup but no finite index nilpotent subgroup.*

3. *There is a* $d \in \mathbb{N}$ *such that*

$$|\partial\Omega| \geq \frac{|\Omega|^{\frac{d-1}{d}}}{C}, \forall\Omega \subset \Gamma,$$

and there is an exhausting family as above with

$$\frac{n}{C} \leq |\Omega_n| \leq Cn,$$

$$\frac{n^{\frac{d-1}{d}}}{C} \leq |\partial\Omega_n| \leq Cn^{\frac{d-1}{d}}$$

$\Leftrightarrow$ *there is a* $d \in \mathbb{N}$ *such that* $p_n(e,e) \leq Cn^{-\frac{d}{2}}$ *and* $p_{2n}(e,e) \geq \frac{n^{-\frac{d}{2}}}{C}$ $\Leftrightarrow \Gamma$ *contains a (finite index nilpotent group)* N *and* $d = \sum_{i=1}^{c} i \times rank(N^i/N^{i+1})$ *where* N^i *is the ith term in the descending central series of* N.

Remark 2.3. The reverse inequality $|\partial\Omega| \leq |\Omega|$ in 1) above is obvious because by definition $\partial\Omega \subset \Omega$. The reverse inequality $p_{2n}(e,e) \geq \frac{\exp(-Cn)}{C}$ in 1) above is true because the number of edge paths in the Cayley graph of Γ with respect to S of length $2n$ which start at e is $|S|^{2n}$ and as $2n$ is even there are loops among them.

According to Tits [47] a subgroup of a Lie group with a finite number of connected components either contains a free subgroup on two generators or contains a co-compact solvable group. According to Mal'cev, polycyclic groups are characterised among solvable ones by the property of having all their abelian subgroups finitely generated [44] 15.2.1. A discrete abelian subgroup of a connected Lie group is finitely generated (take the image under the adjoint representation and consider its Zariski closure which is an algebraic-hence has finitely many connected components-abelian group). According to Mostow, up to finite index, a polycyclic group is a lattice in a simply connected solvable Lie group; see [43] 4.28, hence its growth is either exponential or polynomial [28], [23]. (Another possibility here is to apply the results of Wolf and Milnor on the growth of finitely generated solvable groups [59], [31].) According to Gromov the finitely generated groups with polynomial growth are the virtually nilpotent ones [19] so that their growth is given by the above formula [4]. These facts together explain the algebraic description in the corollary.

The remaining statements in the corollary will follow from the proof of the theorem and from discretisation procedures as in [39]. The crucial point allowing this strategy is the result of Mostow, cited above. With the exception of the upper bounds on the isoperimetric profiles, those results have been proved by several authors, directly in the discrete setting. The lower bounds on the isoperimetric profiles follow from [11]. The wanted upper bound on the heat kernel for groups with exponential growth is established in [57, VII 1.1] and the lower bound for polycyclic groups in [2]. For the heat kernel estimates of the type $n^{-\frac{d}{2}}$; see [57, VI.5] and [26] or [8] for the lower bounds.

2.5 Remarks on "non-lattices"

Even if we do not assume that the finitely generated group Γ is a discrete subgroup in a Lie group, the equivalence of the three conditions (that is the geometric one, the analytic one and the algebraic one) of the classes 1) and 3) remains true, thanks to [14] and [29] for 1) and [57] VI.5 and [19] for 3). See [40] for more on this. Although the groups

$$< a, b : aba^{-1} = b^q >$$

with $q > 1$ are not discrete subgroups in any (real) Lie group with a finite number of connected components, they have

$$\exp(-n^{\frac{1}{3}})$$

as diagonal heat kernel decay and

$$\frac{n}{\log(n)}$$

as isoperimetric profile. An algebraic description of finitely generated groups with $\exp(-n^{\frac{1}{3}})$ as heat kernel decay would be interesting. See [40], [8] for more on this.

3. Preliminary material

3.1 Unimodular Lie groups

Among Lie groups (with left-invariant Riemannian metrics), unimodularity is not a quasi-isometric invariant. For example a simple Lie group $G = KAN$ (with finite center) is quasi-isometric to the solvable part AN of an Iwasawa decomposition. If the real rank of G is non-zero then AN is non-unimodular. But the strong isoperimetric inequality implied by the non-unimodularity is invariant under quasi-isometries. We recall in the context of Lie groups well known properties of unimodularity (see for example [43, Ch.I], [45, Ch.10]). The following result is standard.

Proposition 3.1. *Let G be a connected Lie group. Then G is unimodular if and only if for any left-invariant vector field X on it $divX = 0$. More precisely,*

$$divX = -\frac{d}{dt}\Delta(\exp tX)|_{t=0},$$

where

$$\Delta : G \to \mathbb{R}_+^*$$

is the modular homomorphism. Hence $divX = 0$ if and only if $X(e)$ belongs to the kernel of the derivative $T_e\Delta$.

Corollary 3.1 (compare [57, Ch. 9]). *Let G be a connected Lie group which is non-unimodular. Then for any left-invariant Riemannian metric on G there is a constant $\epsilon > 0$ such that, for all domains Ω in G with regular boundary,*

$$vol(\Omega) \geq \epsilon \times vol(\partial\Omega).$$

Proof. Let X be a left-invariant vector field on G of unit norm such that $divX = \epsilon > 0$. Let ω be the left-invariant volume form on G. Stokes formula shows that

$$\begin{aligned} \epsilon \times vol(\Omega) &= \int_\Omega divX\omega \\ &= \int_{\partial\Omega} i_X\omega \leq vol(\partial\Omega). \end{aligned}$$

q.e.d.

Proposition 3.2. *(see [43] I.1.4) Let G be a connected Lie group and let H be a closed subgroup. Then there exists an invariant volume form on the G-space G/H if and only if the restriction of the modular homomorphism of G to H is equal to the modular homomorphism of H.*

Corollary 3.2. *Let G be a connected Lie group. Let H be a closed subgroup. If H is normal and if G is unimodular then H is unimodular. If H is co-compact and unimodular then G is unimodular.*

Proof. If H is normal the G-homogeneous space G/H is a Lie group hence there is a G-invariant volume form on it. If H is co-compact then G is an H-principal bundle with compact base B. Let $B = \cup K_i$ be a finite cover of B by compact sets K_i such that the bundle restricted to each K_i is trivial. Let $s_i : K_i \to G$ be continuous sections. The subset $S = \cup s_i(K_i)$ is compact in G and each element $g \in G$ can be written as $g = sh$ with $s \in S$ and $h \in H$. If H is unimodular then $\Delta_G(G) = \Delta_G(S)$ is a compact subgroup of $\mathbb{R}^*_+$. q.e.d.

Proposition 3.3. *Let G be a connected Lie group and let N be a closed normal subgroup. Let $\pi : G \to G/N$ be the projection. If the action by conjugation of G on N preserves a volume form on N, then $\Delta_{G/N} \circ \pi = \Delta_G$.*

Proof. If a non-zero top-dimensional differential form on a Lie group is left-invariant, we call it a volume form. Let $i : N \to G$ be the inclusion. Let β be a left-invariant form on G such that $i^*\beta$ is a volume form on N. Let α be a volume form on G/N. Then $\pi^*\alpha \wedge \beta$ is a volume form on G. Let $g \in G$. Then

$$\begin{aligned}\Delta_G(g)(\pi^*\alpha \wedge \beta) =& c_g^*(\pi^*\alpha \wedge \beta) = (\pi c_g)^*\alpha \wedge c_g^*\beta \\ =& (c_{\pi(g)}\pi)^*\alpha \wedge c_g^*\beta = \pi^* c^*_{\pi(g)}\alpha \wedge c_g^*\beta \\ =& \pi^*\Delta_{G/N}\pi(g)\alpha \wedge c_g^*\beta = \Delta_{G/N}\pi(g)(\pi^*\alpha \wedge c_g^*\beta) \\ =& \Delta_{G/N}\pi(g)(\pi^*\alpha \wedge \beta).\end{aligned}$$

q.e.d.

Corollary 3.3. *Let G be a connected Lie group and let K be a compact normal subgroup. Then G is unimodular if and only if G/K is unimodular.*

Proof. The above proposition applies because the connected component of $Aut(K)$ is compact (the proof that this group is compact is classical, the main point being that the component of the identity of the automorphisms group of a semi-simple Lie algebra contains only inner automorphisms [48] 3.10.8). q.e.d.

3.2 Stability of the heat kernel

The techniques explained in [39] apply to show the following. Let X and Y be connected complete Riemannian manifolds which are quasi-homogeneous (see Theorem 2.1 for the definition). If X is quasi-isometric to Y then

$$p_t(x,x) \sim p_t(y,y).$$

3.3 Riemannian submersions

The upper bounds on the isoperimetric profiles given by exhaustions are preserved if we pass from the base of a Riemannian submersion with finite constant volume fibers to the total space. Specifically, we will need the following statement which is a straigthforward application of the co-area formula.

Proposition 3.4. *Let X and Y be connected Riemannian manifolds and let $p : X \to Y$ be a Riemannian submersion. Assume that the volume of the fibers is constant equal to c. Then the following is true:*

1. *If W is a submanifold (with corners) of Y and $V = p^{-1}(W)$, then $vol(V) = c \times vol(W)$ and $vol(\partial V) = c \times vol(\partial W)$.*

2. *If W_t is a continuous family of submanifolds (with corners) in Y such that $W_t \subset W_s$ if $t \leq s$ and $Y = \bigcup_{t\geq 0} W_t$, then the family $V_t = p^{-1}(W_t)$ has the same properties in X.*

Such submersions arise via group theory as follows.

Proposition 3.5. *Let G be a connected Lie group, let K be a compact subgroup and let O be a closed subgroup of K. For any G-invariant Riemannian metric on G/K, there exists a G-invariant Riemannian metric on G/O such that the projection*

$$p : G/O \to G/K$$

is a Riemannian submersion.

Proof. The adjoint action of K on T_eG stabilizes T_eK and as K is compact we can find a stable complement H_e. Let $p : G \to G/K$ be the projection. The restriction of

$$T_ep : H_e \oplus T_eK \to T_e(G/K)$$

to H_e is an isomorphism. On H_e we choose the scalar product which makes this isomorphism an isometry. This scalar product is $Ad(K)$-invariant because if $X \in H_e$ and $k \in K$,

$$\begin{aligned} ||T_eAd(k)X||_e &= ||T_epT_eAd(k)X||_{p(e)} \\ &= ||T_k(pr_{k^{-1}})T_el_kX||_{p(e)} = ||T_kpT_el_kX||_{p(e)} \\ &= ||T_{p(e)}l_kT_epX||_{p(e)} = ||T_epX||_{p(e)} = ||X||_e. \end{aligned}$$

(We have denoted also by l_g the action of g on G/K.) We choose a scalar product on T_eK which is also $Ad(K)$-invariant to get an $Ad(K)$-invariant scalar product on the whole space $T_eG = H_e \oplus T_eK$. It induces a left-invariant Riemannian metric on G and as p is G-equivariant, it is a Riemannian submersion. Moreover, this Riemannian metric, being left-invariant under G and right-invariant under K, induces a G-invariant Riemannian metric on G/O such that the projection

$$\pi : G \to G/O$$

is a Riemannian submersion. As p and π are Riemannian submersions, the quotient map

$$q : G/O \to G/K$$

induced by p is also a Riemannian submersion. q.e.d.

3.4 From amenable to solvable simply connected

Bilipschitz transformations are quasi-isometries. They obviously preserve the asymptotic behavior of the isoperimetric profiles and the exhaustions. To handle amenable groups we will use the following.

Proposition 3.6. *Let G be a connected Lie group and let R be its radical. Assume that G/R is compact (i.e., that G is amenable). Let $K \subset G$ be a maximal compact subgroup and consider a G-invariant metric on the homogeneous space G/K. Then there exist a simply connected solvable Lie group S and a compact homogeneous space B such that G/K is bilipschitz equivalent to $S \times B$ endowed with the product metric of any left-invariant Riemannian metric on S and any Riemannian metric on B. Moreover, G is unimodular if and only if S is unimodular.*

The first step in order to prove the above proposition is the following.

Proposition 3.7. *Let R be a connected solvable Lie group. There is a compact connected central subgroup $T \subset R$ such that the quotient R/T contains a closed normal co-compact subgroup which is connected and simply connected.*

Proof. Let $\pi : G \to R$ be the universal cover of R and let $\Gamma \simeq \pi_1(R)$ be the corresponding discrete central Galois subgroup of G. Let N be the closure of the commutator subgroup of R. This is a connected nilpotent group (see [48] Th. 3.18.8 and [27] II.4.1 Cor. C). The identity component H of $\pi^{-1}(N)$ contains the commutator subgroup of G, hence we have two exact sequences with commutative squares:

$$\begin{array}{ccccccccc} 1 & \to & H & \to & G & \to & A & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & N & \to & R & \to & B & \to & 1 \end{array}$$

with A and B abelian and connected. As G is simply connected and H is connected, A is simply connected hence isomorphic to a vector space. Viewing G as a H-principal bundle with contractible basis A we deduce that H is also simply connected. As H is nilpotent and simply connected, its center Z is isomorphic to a vector space. Let E be the vector subspace of Z generated by $\Gamma \cap Z$. As Z is a characteristic subgroup of H and as H is normal in G and Γ central in G , we deduce that E is central in G. Let Λ be the subgroup generated by E and Γ. It is a closed central subgroup of G with identity component equal to E and the quotient

$$T = \Lambda/\Gamma$$

is a central connected compact Lie group. (To check that Λ is closed, notice that it is contained in the closed subgroup $\pi^{-1}(N)$.) There is an exact sequence

$$1 \to \Lambda/\Gamma \to G/\Gamma \to G/\Lambda \to 1.$$

As $R = G/\Gamma$ we are looking for a simply connected co-compact subgroup in G/Λ. We have two exact sequences with commutative squares:

$$\begin{array}{ccccccccc} 1 & \to & H & \to & G & \to & A & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H/E & \to & G/\Lambda & \to & B & \to & 1. \end{array}$$

Notice that the group H/E is simply connected. We denote by

$$p : G/\Lambda \to B$$

the projection. Let $X \subset B$ be a simply connected closed subgroup of maximal dimension. Hence X is isomorphic to a vector space and B/X is compact. The subgroup $p^{-1}(X) \subset G/\Lambda$ is closed normal and co-compact, and we can view it as a H/E-principal bundle with contractible basis X. Hence it is simply connected. q.e.d.

We will need the following fact.

Proposition 3.8. *Let G be a connected Lie group. Let H be a normal subgroup. Let T be a compact connected subgroup of H. If T is central in H then T is central in G.*

Proof. The closure of H in G is normal and T is central in the closure of H. Hence we can assume that H is closed. Let Z be the connected component of the identity of the center of H. Let $\Gamma \simeq \pi_1(Z)$. An automorphism of Z induces an automorphism of the universal cover $\tilde{Z}$ of Z which preserves the Galois group $\Gamma \subset \tilde{Z}$. We get a continous map

$$G \to Aut(\tilde{Z})$$

$$g \mapsto \tilde{c_g}$$

where $\tilde{c_g}$ is the lift of the automorphism of Z given by the conjugation by g. As Γ is discrete in $\tilde{Z}$ and as G is connected we get for each $g \in G$

$$\tilde{c_g}(\gamma) = \gamma, \forall \gamma \in \Gamma.$$

Let $E \subset \tilde{Z}$ be the vector subspace generated by Γ. We have

$$\tilde{c_g}(z) = z, \forall z \in E.$$

Hence

$$c_g(z) = z, \forall z \in E/\Gamma.$$

Let $\Lambda \simeq \pi_1(T)$. By hypothesis $T \subset Z$ so that $\tilde{T} \subset \tilde{Z}$ and

$$T = \tilde{T}/\Lambda \subset E/\Gamma.$$

q.e.d.

Proposition 3.9. *Let $1 \to S \to G \to K \to 1$ be an exact sequence of connected Lie groups with S solvable and K compact. The principal S-bundle G is isomorphic to the trivial bundle $K \times S$.*

Proof. As the universal cover of a solvable Lie group is contractible, the only obstruction to the existence of a section lies in $\pi_2(K)$ which is trivial. q.e.d.

Let G and S be as above. The following lemma implies that a left-invariant metric on G is bilipschitz equivalent to a product metric on $S \times G/S$.

Lemma 3.1. *Let E be a (left) principal H-bundle. If the bundle is trivial and the basis is compact, then any two H-invariant Riemannian metrics on E are bilipschitz equivalent.*

Proof. We fix a left-invariant Riemannian metric on H and a Riemannian metric on B and we consider the corresponding product metric on $H \times B$. Let us show that any H-invariant Riemannian metric on E is bilipschitz equivalent to this product metric. Let $\sigma : B \to E$ be a section. Let $\Psi : H \times B \to E$ be the trivialisation of E defined by $\Psi(h, b) = h\sigma(b)$. We denote by l_h the left action of h on $H \times B$ and on E. There is a commutative diagram

$$\begin{array}{ccc} H \times B & \xrightarrow{\Psi} & E \\ {\scriptstyle l_h}\downarrow & & \downarrow{\scriptstyle l_h} \\ H \times B & \xrightarrow{\Psi} & E. \end{array}$$

Let $p = (h, b) \in H \times B$. Taking the derivatives we get the commutative diagram

$$\begin{array}{ccc} T_{(e,b)}(H \times B) & \xrightarrow{T_{(e,b)}\Psi} & T_{\sigma(p)}E \\ {\scriptstyle T_{(e,b)}l_h}\downarrow & & \downarrow{\scriptstyle T_{\sigma(b)}l_h} \\ T_p(H \times B) & \xrightarrow{T_p\Psi} & T_{\Psi(p)}E. \end{array}$$

Hence

$$\|T_p\Psi\| \leq \sup_{b \in B} \|T_{(e,b)}\Psi\|$$

$$\|T_p\Psi\| \geq \inf_{b \in B} \|T_{(e,b)}\Psi\|.$$

q.e.d.

Proposition 3.10. *Let $G = NK$ be a semi-direct product of connected Lie groups with K compact. Let $p : G \to N$ be the map defined* *by $p(nk) = n$. Assume that there is a subgroup $S \subset G$ such that the restriction of p to S is a diffeomorphism. Then S and N (equipped with left-invariant Riemannian metrics) are bilipschitz equivalent.*

Proof. On G we forget for a while the group structure defined by the given semi-direct product NK and replace it by the direct product structure $N \times K$. The projection p becomes a homomorphism and its restriction to the subgroup S is an isomorphism. But an isomorphism between two Lie groups (with left-invariant Riemannian metrics) is a bilipschitz transformation. According to Lemma 3.1, a left-invariant Riemannian metric on G for its original group structure is bilipschitz equivalent to a left-invariant Riemannian metric for the direct product structure. q.e.d.

We prove Proposition 3.6.

Proof. Let r be the Lie algebra of the radical R of G. Let

$$g = r + m$$

be a Levi decomposition of the Lie algebra g of G. As the quotient G/R is compact by hypothesis, the semi-simple group M associated to m is compact. Hence $G = RM$ with $R \cap M$ finite (see [48] Th.3.18.13). As the maximal compact subgroups of G are conjugate (see [25] XV.3.Th.3.1) we can choose the above Levi decomposition such that M is contained in the given maximal compact subgroup K. Applying Proposition 3.7 to R we get a compact connected central subgroup T in R and a simply connected solvable group S which is normal and co-compact in R/T. According to Corollary 3.2 and Corollary 3.3 the group G is unimodular if and only if S is unimodular. According to Proposition 3.8 the group T is also central in G hence T is contained in any maximal compact subgroup. In particular, the quotient R/T acts (properly by isometries on the left) on $G/K = X$. The action of $S \subset R/T$ on X is free because in a simply connected solvable Lie group the only compact subgroup is the trivial one. Hence we get a (left) principal S-bundle $X \to B$. As $G = RM$ and $M \subset K$ the action of R/T on X is transitive. This implies that the action of the quotient group Q of R/T by S is also transitive on $B = X/S$. By hypothesis Q is compact. Hence B is a compact homogeneous space. A simply connected solvable group is contractible hence the principal bundle is trivial. Applying Lemma 3.1 completes the proof.

4. From homogeneous Riemannian manifolds to simply connected solvable Lie groups

With this preliminary material in mind we are ready to reduce the proofs of Theorem 2.1 and of Proposition 2.1 to the case of simply connected solvable Lie groups.

4.1 The theorem of Myers and Steenrod

Let X be a connected homogeneous Riemannian manifold and let G be its group of isometries. Myers and Steenrod [33] have shown that G is a Lie group and that the map

$$G \to X$$

$$g \mapsto gx_0$$

where x_0 is a base point is differentiable. Hence we get a diffeomorphism $G/K \simeq X$ where $K \subset G$ is the isotropy group of x_0. As we assume, X is connected and $G \to G/K$ is a fibration $G^0/G^0 \cap K \simeq X$ where G^0 is the connected component of the identity in G. Denoting again by G this connected component and applying Proposition 3.5 we get a G-equivariant Riemannian submersion $G \to X$ with compact fiber. In particular X is quasi-isometric to G with a left-invariant Riemannian metric.

4.2 Non-unimodularity and non-amenability for Lie groups

On a connected Lie group G the Riemannian Laplacian (associated with a left-invariant Riemannian metric) has a spectral gap if and only if the corresponding heat kernel satisfies

$$p_t(e,e) \sim \exp(-t).$$

In terms of the Rayleigh quotient, the spectral gap is equivalent to the Sobolev inequality

$$||df||_2 \geq \epsilon ||f||_2,$$

where $\epsilon > 0$ is a constant and f is any smooth function on G with compact support. On G the local structure is the same at every point hence the local Poincaré inequality and the local doubling volume property hold (see [50]). It follows that the above L_2 inequality is equivalent to the following L_1 inequality

$$\|df\|_1 \geq \epsilon \|f\|_1;$$

see [12]. This last inequality is equivalent to the strong isoperimetric inequality

$$vol(\partial\Omega) \geq \epsilon \times vol(\Omega),$$

where Ω is any measurable domain with regular boundary. A unimodular connected Lie group G is non-amenable if and only if the Riemannian Laplacian has a spectral gap (see for example [55]). In this case, applying Proposition 3.4 to the submersion $G \to X$ we get $I_X(t) \succeq t$. Should G be non-unimodular, we apply Corollary 3.1 and deduce in the same way $I_X(t) \succeq t$.

4.3 Bishop's theorem

A homogeneous Riemannian manifold X has bounded curvature, hence we can consider the exhaustion of X by the balls centered at $x_0 \in X$ and apply Bishop's theorem [16] 4.19, to deduce that $I_X(t) \preceq t$. The submersion $G \to X$ is a quasi-isometry, hence the asymptotic behavior of the heat kernel on X and on G are the same. Together with 4.1, 4.2, 4.3 above, this explains the equivalences of the theorem in the case of non-amenable groups and in the case of non-unimodular groups.

4.4 Amenable unimodular Lie groups

If G is amenable, according to Furstenberg [60] 4.1.9, the semi-simple quotient G/R where R is the radical is a compact group. According to Proposition 3.6 and Proposition 3.5, the homogeneous space X is the total space of a G-equivariant Riemannian submersion with compact fiber and with basis bilipschitz equivalent to a Riemannian product $S \times B$ where S is a simply connected solvable Lie group which is unimodular if and only if G has this property, and where B is a compact homogeneous space. The lower bounds on the isoperimetric profile follow from the inequality of Coulhon and Saloff-Coste [11]. Notice that the hypothesis in their result involves the growth of the balls and that this growth is invariant under quasi-isometries. According to Auslander and Green [1], to each simply connected solvable Lie group S corresponds naturally a simply connected nilpotent Lie group N. Should S have polynomial growth, it is bilipschitz equivalent to a simply connected nilpotent Lie group. To prove it, we use the fact that a group of polynomial growth is of type R [28]. Hence, according to Theorem 4.2 of [1] the abelian semi-simple group T associated to S is compact. According to Theorem 4.1 of [1] the projection $p : NT \to N$ from the semi-simple splitting of S onto N is a diffeomorphism when restricted to $S \subset NT$. Applying Proposition 3.10 completes the proof. Hence $p_t(e,e) \sim t^{-\frac{d}{2}}$ where d is the degree of growth of N. See [57]. If the growth of S is not polynomial then S has exponential growth [23], [28] and $p_t(e,e) \sim \exp(-t^{\frac{1}{3}})$; see [24], [56]. The rest of the paper is devoted to the construction of exhaustions for simply connected solvable unimodular Lie groups and their lattices. The exhaustions on X are then obtained by applying Proposition 3.4 to the Riemannian submersions $X \to S \times B$ and $S \times B \to S$.

5. Nilpotent Lie groups

5.1 The Lie algebra as a Lie group

Let N be a simply connected nilpotent Lie group of class c. We denote by L its Lie algebra. Let $L^1 = L, L^{k+1} = [L, L^k]$ be the ideals of the descending central series. We choose supplementary subspaces L_k such that $L^k = L_k \oplus L^{k+1}$ and we choose a norm on L which makes the decomposition $L = \oplus_{k=1}^c L_k$ orthogonal. If $x \in L$ we write $x = \sum_{k=1}^c x_k$ its orthogonal decomposition. We denote by $|x|$ the norm of x. *We transport, via the exponential map, the Lie group structure of N to the Lie algebra L of N.* Hence for $x, y \in L$ the multiplication law is given by the Campbell-Haussdorf-Dynkin formula:

$$xy = x + y + \frac{1}{2}[x,y] + \dots .$$

5.2 Comparing the left-invariant metric with the Euclidean one

We can consider L as an Euclidean space with an orthogonal decomposition $L = L_1 \oplus \dots \oplus L_c$. We can also consider L as a Lie group with the left-invariant Riemannian metric obtained by translating the above Euclidean metric on $T_0 L \simeq L$. If $h \in T_x L$ we denote by $|h|_x$ its Riemannian norm. Let $x \in L$. As the left translation by x is given by the

C.H.D. formula

$$l_x(y) = x + y + \frac{1}{2}[x, y] + \dots$$

we see that its derivative at 0 is

$$T_0 l_x = id + \frac{1}{2} ad(x) + \dots + cst \times ad^{c-1}(x),$$

where cst is a constant, $ad(x)(y) = [x, y]$, and $ad^j(x)$ denotes $ad(x)$ composed j times. Let $h \in L$. Writing $h = \sum_{i=1}^c h_i$ with $h_i \in L_i$ we have obviously

$$T_0 l_x(h)_1 = h_1.$$

The following lemma gives an upper bound for the Euclidean norm of $T_0 l_x(h)_k$ when $k > 1$ in terms of the Euclidean norms of x and h. We use the following notation. For $x \in L$, $x = \sum_{i=1}^c x_i$ we define $e_i(x) = 0$ if $x_i = 0$ and $e_i(x) = x_i/|x_i|$ if this is not the case. We put $t_i = |x_i|$. Hence $x = \sum_{i=1}^c t_i e_i(x)$.

Lemma 5.1. *With the notation as above and if the class c is bigger, than 1, then $\forall k \in \{2, \dots, c\}$, $\forall x \in L$, $\forall h \in L$ we have*

$$|T_0 l_x(h)_k| \leq |h_k| + C \sum_{j=1}^{k-1} \sum_{i_1 + \dots + i_{j+1} \leq k} t_{i_1} \dots t_{i_j} |h_{i_{j+1}}|.$$

Proof. In the formula for the derivative of the left-translation by x

$$T_0 l_x = id + \sum_{j=1}^{c-1} c_j ad^j(x),$$

consider $A = \max_j |c_j|$. The map

$$L \times \dots \times L \to \mathbb{R}_+^*$$

$$(z_1, \dots, z_{c-1}) \mapsto \|ad(z_1) \dots ad(z_{c-1})\|_{End(L)}$$

defined on the direct product of $c - 1$ copies of L is continuous. Hence when restricted to the direct product of $c - 1$ copies of the unit ball in L (for the Euclidean norm), it is bounded by a constant B. So

$$T_0 l_x(h)_k = h_k + \sum_{j=1}^{k-1} c_j ad^j(x)(h)_k,$$

and we have

$$|T_0 l_x(h)_k| \leq |h_k| + A \sum_{j=1}^{k-1} |ad^j(x)(h)_k|.$$

But

$$ad^j(x)h = \sum_{i_1,...,i_{j+1}} t_{i_1}...t_{i_j} ad(e_{i_1}(x))...ad(e_{i_j}(x))(h_{i_{j+1}}),$$

hence

$$\begin{aligned}|ad^j(x)(h)_k| \leq & \sum_{i_1+...+i_{j+1}\leq k} t_{i_1}...t_{i_j} |ad(e_{i_1}(x))...ad(e_{i_j}(x))(h_{i_{j+1}})| \\ \leq & B \sum_{i_1+...+i_{j+1}\leq k} t_{i_1}...t_{i_j} |h_{i_{j+1}}|.\end{aligned}$$

Choose $C = AB$. q.e.d.

Corollary 5.1. *With the notation as above, there exists a constant $C > 1$ such that $\forall k \in \{1, ..., c\}$, $\forall t \geq 1$. If $x \in L$ is such that $t_i \leq t^i$ for $i \leq k-1$ and if $h \in L$ is such that $|h| \leq 1$, then*

$$|T_0 l_x(h)_k| \leq C t^{k-1}.$$

Lemma 5.2. *With the notation as above, there exists a constant $D > 1$ such that if $t \geq 1$, if $x \in L$ is such that $t_i \leq t^i$ for $1 \leq i \leq c$ and if $c : [0, \epsilon] \to L$ is a differentiable path parametrised by the Riemannian arc length with $c(0) = x$ then*

$$|c(\epsilon)_i| \leq (t + \epsilon D)^i.$$

Proof. If the class c is bigger than 1, let $1 \leq k \leq c-1$. (If $c = 1$ we put k=1.) We have

$$|c(\epsilon)_k| \leq |c(0)_k| + \int_0^\epsilon |\frac{d}{ds} c(s)_k| ds.$$

As $|\frac{d}{ds} c(s)|_{c(s)} = 1$, by definition of the left-invariant Riemannian metric, there exists $h \in T_0 L$ (depending on s) such that $|h| = 1$ and such that

$$T_0 l_{c(s)} h = \frac{d}{ds} c(s).$$

As already mentioned $T_0 l_{c(s)}(h)_1 = h_1$. Hence $|\frac{d}{ds}c(s)_1| = |h_1| \leq 1$ and $|c(\epsilon)_1| \leq t + \epsilon$. We assume now that the inequality of the lemma is true for $i \leq k$ and we will show that the inequality is true for $k+1$. Applying the induction hypothesis to the restrictions of the path c to $[0, s]$ where $0 \leq s \leq \epsilon$, we obtain, if $i \leq k$, the inequality $|c(s)_i| \leq (t + \epsilon D)^i$. Corollary 5.1 implies that

$$|\frac{d}{ds}c(s)_{k+1}| \leq C(t + \epsilon D)^k.$$

Hence

$$|c(\epsilon)_{k+1}| \leq t^{k+1} + \epsilon C(t + \epsilon D)^k \leq (t + \epsilon M)^{k+1},$$

where M is a constant depending on k, C and D only. As the induction process involves finitely many steps, the lemma is proved. q.e.d.

6. Derivatives and commutators

6.1 A universal commutator

Let G be a group. Let $r \in \mathbb{N}$. Let $a_1, ..., a_r \in G, b_1, ..., b_r \in G$. We define for $j \in \mathbb{N}$, $P_j = \prod_{i=j}^{r} a_i$ if $j \leq r$ and $P_j = e$ if $j > r$. We define also $Q_j = \prod_{i=1}^{j} a_i b_i$ and $Q_0 = e$. In groups we use the notation $[x, y]$ for the commutator $xyx^{-1}y^{-1}$ and $c_x(y) = xyx^{-1}$ for the conjugate of y by x.

Lemma 6.1. *With the above notation,*

$$a_1...a_r b_1...b_r = (\prod_{j=1}^{r} [a_j, P_{j+1}Q_{j-1}])a_1 b_1 a_2 b_2 ... a_r b_r.$$

Proof. The equality of the lemma is obtained by putting $i = 0$ in the following formula:

$$P_{i+1}Q_i b_{i+1}...b_r = (\prod_{j=1}^{r-i} [a_{j+i}, P_{j+1+i}Q_{j-1+i}])Q_r.$$

If $i = r - 1$ the above equality reduces to

$$a_r Q_{r-1} b_r = [a_r, Q_{r-1}]Q_r,$$

which is easy to check. We assume the formula is true for $1 \leq i \leq r-1$ and we want to show that it is true for $i-1$. It is easy to check that

$$P_i Q_{i-1} b_i ... b_r = [a_i, P_{i+1} Q_{i-1}] P_{i+1} Q_i b_{i+1} ... b_r.$$

Applying the induction hypothesis to $P_{i+1} Q_i b_{i+1} ... b_r$ completes the proof.

q.e.d.

For a given family of elements $a_1, ..., a_r$ we may consider the commutator

$$\prod_{j=1}^{r} [a_j, P_{j+1} Q_{j-1}]$$

of the above lemma as a map

$$C(a_1, ..., a_r) : G^r \to G.$$

Notice that the equality of the lemma shows that

$$C(a_1, ..., a_r)(e, ..., e) = e.$$

6.2 The derivative of the commutator in $Gl(n, \mathbb{R})$

Let $G = Gl(n, \mathbb{R})$ (we use capital scripts $A_1, ..., A_r$) and we denote the identity element by id. We need a bound on the norm of the derivative

$$T_{(id,...,id)} C(A_1, ..., A_r) : M(n, \mathbb{R})^r \to M(n, \mathbb{R})$$

in terms of the norms of the A_i. If $M \in M(n, \mathbb{R})$, we denote by $\|M\| = \sup_{|x| \leq 1} |M(x)|$ its operator norm.

Lemma 6.2. *Let $r \geq 2$, let $H \in M(n, \mathbb{R})^r$, and $H = \sum_{i=1}^{r} H_i$. Then*

$$\|T_{(id,...,id)} C(A_1, ..., A_r)(H)\| \leq 2(r-1)^2 \sup_{1 \leq i \leq r} \|A_i^{\pm 1}\|^{6r+2} \sup_{1 \leq i \leq r} \|H_i\|.$$

Proof. The function from $\mathbb{R}$ to $M(n, \mathbb{R})$ defined by

$$t \mapsto C(A_1, ..., A_r)(id + tH_1, ..., id + tH_r)$$

is analytic in t with coefficients in $M(n, \mathbb{R})$. In order to compute the coefficient of t, we write

$$(id + tH_k)^{-1} = id - tH_k + t^2 H_k^2 - t^3 H_k^3 + ...$$

and compute the coefficient of t in the expression

$$[A_1, P_2]\prod_{j=2}^{r} A_j P_{j+1}(\prod_{k=1}^{j-1} A_k(id+tH_k))A_j^{-1}(\prod_{k=1}^{j-1}(id-tH_{j-k})A_{j-k}^{-1})P_{j+1}^{-1}A_j^{-1}.$$

For $2 \leq j \leq r$ let c_j^0 be the constant coefficient and let c_j^1 be the coefficient of t in

$$A_j P_{j+1}(\prod_{k=1}^{j-1} A_k(id + tH_k))A_j^{-1}(\prod_{k=1}^{j-1}(id - tH_{j-k})A_{j-k}^{-1})P_{j+1}^{-1}A_j^{-1}.$$

Then we have

$$c_j^0 = (A_j...A_r A_1...A_{j-1})A_j^{-1}(A_j...A_r A_1...A_{j-1})^{-1}$$

and

$$c_j^1 = (A_j...A_r)(\sum_{k=1}^{j-1} A_1...A_k H_k...A_j)(A_1...A_{j-1})^{-1}(A_j...A_r)^{-1}$$
$$- (A_j...A_r)(A_1...A_{j-1})(\sum_{k=1}^{j-1} A_j^{-1}...H_k A_k^{-1}...A_1^{-1})(A_j...A_r)^{-1}.$$

We denote by $i_2, ..., i_r$ indices with values in $\{0, 1\}$:

$$T_{(id,...,id)}C(A_1, ..., A_r)(H) = [A_1, P_2] \sum_{i_2+...+i_r=1} c_2^{i_2}...c_r^{i_r}.$$

Therefore

$$\begin{aligned}
&\|T_{(id,...,id)}C(A_1, ..., A_r)(H)\| \\
&\quad\leq \sup_{1\leq i\leq r} \|A_i^{\pm 1}\|^{2r}(r-1) \sup_{i_2+...i_r=1} \|c_2^{i_2}\|...\|c_r^{i_r}\| \\
&\quad\leq \sup_{1\leq i\leq r} \|A_i^{\pm 1}\|^{2r}(r-1) \sup_{2\leq j\leq r} \|c_j^1\| \sup_{2\leq j\leq r} \|c_j^0\|^{r-2}.
\end{aligned}$$

As

$$\|c_j^0\| \leq \sup_{1\leq i\leq r} \|A_i^{\pm 1}\|^{2r+1}$$

and

$$\|c_j^1\| \leq 2(r-1) \sup_{1\leq i\leq r} \|H_i\| \sup_{1\leq i\leq r} \|A_i^{\pm 1}\|^{2r+1},$$

the inequality of the lemma is now immediate. q.e.d.

We recall the following fact which is an obvious consequence of the inequality $\|AB\| \leq \|A\|\|B\|$. For $t \in \mathbb{R}$ we denote by $[t]$ its integral value.

Lemma 6.3. *Let $\phi : \mathbb{R} \to Gl(n, \mathbb{R})$ be a homomorphism. Let $C \geq 1$ and assume that $\|\phi(t)\| \leq C$ if $|t| \leq 1$. Then $\|\phi(t)\| \leq C^{|[t]|+1}$.*

6.3 The derivative of the commutator in Lie groups

Let G be a Lie group. We choose a norm on T_eG. If $X \in T_eG$, we denote its norm by $|X|$.

Corollary 6.1. *If G is a simply connected Lie group, $r \in \mathbb{N}$, then there exists a constant $\alpha > 1$ such that if $X_1, ..., X_r \in T_eG$ with $|X_i| \leq 1$, if $a_1, ..., a_r \in \mathbb{R}$ and if $H \in T_eG^r, H = \sum_{i=1}^r H_i$ then*

$$|T_{(e,...,e)}C(\exp(a_1X_1), ..., \exp(a_rX_r))H| \leq \alpha \sup_{1\leq i\leq r} \alpha^{|a_i|} \sup_{1\leq i\leq r} |H_i|.$$

Proof. Let T_eG be the Lie algebra of G. Thanks to Ado's theorem (see for example [48]), it embeds into the Lie algebra $M(n, \mathbb{R})$ of the Lie group $Gl(n, \mathbb{R})$ for some n. The embedding of Lie algebras induces a morphism of local Lie groups. By hypothesis G is simply connected hence the morphism of local Lie groups extends to a homomorphism of Lie groups

$$\phi : G \to Gl(n, \mathbb{R})$$

with derivative at e

$$T_e\phi : T_eG \to M(n, \mathbb{R})$$

given by the above embedding. Notice that

$$\delta = \min_{H\in T_eG, |H|=1} \|T_e\phi(H)\|$$

is non-zero. Let

$$\sigma : \mathbb{R}^r \to G^r$$

$$\sigma(a_1, ..., a_r) = (\exp(a_1X_1), ..., \exp(a_rX_r)),$$

and let

$$\phi^r : G^r \to Gl(n, \mathbb{R})^r$$

be the obvious product map induced by ϕ. We have the commutative square

$$\begin{array}{ccc} G^r & \xrightarrow{C(\sigma(a))} & G \\ \phi^r\downarrow & & \downarrow\phi \\ Gl(n,\mathbb{R})^r & \xrightarrow{C(\phi^r(\sigma(a)))} & Gl(n,\mathbb{R}). \end{array}$$

Taking derivatives, we get the commutative square

$$\begin{array}{ccc} (T_eG)^r & \xrightarrow{T_{(e,\dots,e)}C(\sigma(a))} & T_eG \\ T_e\phi^r\downarrow & & \downarrow T_e\phi \\ M(n,\mathbb{R})^r & \xrightarrow{T_{(id,\dots,id)}C(\phi^r(\sigma(a)))} & M(n,\mathbb{R}). \end{array}$$

Hence

$$|T_{(e,\dots,e)}C(\sigma(a))H| \leq \delta^{-1}\|T_{(id,\dots,id)}C(\phi^r(\sigma(a)))(T_e\phi H_1,\dots,T_e\phi H_r)\|.$$

Applying Lemma 6.2, we see that this number is less then or equal to

$$\delta^{-1}c\sup_{1\leq i\leq r}\|\phi(\exp(\pm a_iX_i)\|^c\sup_{1\leq i\leq r}\|T_e\phi H_i\|,$$

where $c=\max(2(r-1)^2,6r+2)$. The corollary now follows from Lemma 6.3.

7. Elementary inequalities

We will need the following elementary inequalities.

Lemma 7.1. *Let $\lambda\geq\mu\geq\nu>0$, $n\in\mathbb{N}$, and let $t\geq\epsilon'\geq\epsilon\geq 0$ such that $\lambda^t-\epsilon'\mu^t\geq 0$. Then*

$$(\lambda^t-\epsilon'\mu^t)^n+(\epsilon'-\epsilon)\nu^t(\lambda^t-\epsilon\mu^t)^{n-1}\leq(\lambda^t-\epsilon\mu^t)^n.$$

Proof. If $\lambda^t-\epsilon\mu^t=0$ it is obvious. If not,

$$(\lambda^t-\epsilon'\mu^t)(\frac{\lambda^t-\epsilon'\mu^t}{\lambda^t-\epsilon\mu^t})^{n-1}+(\epsilon'-\epsilon)\nu^t\leq\lambda^t-\epsilon'\mu^t+(\epsilon'-\epsilon)\nu^t\leq\lambda^t-\epsilon\mu^t$$

and we multiply by $(\lambda^t-\epsilon\mu^t)^{n-1}$. q.e.d.

Lemma 7.2. *Let $d,r\geq 1$ and let $\lambda>\mu>0$, let $\epsilon\in\mathbb{R}$. Then there exists $C>1$ such that if $t\geq 1$ then*

$$(\lambda^t-(\epsilon-1)\mu^t)^d(t-(\epsilon-1))^r-(\lambda^t-(\epsilon+1)\mu^t)^d(t-(\epsilon+1))^r\leq C(\lambda^t)^dt^{r-1}.$$

Proof. The function $f(x, y) = x^d y^r$ is C^1 hence, given $K > 0$, there exists $C > 1$ such that if $p, q \in \mathbb{R}, |p| < K, |q| < K$ then $|f(p) - f(q)| \leq C|p - q|$. Hence we obtain $C > 1$ such that

$$(1-(\epsilon-1)(\mu/\lambda)^t)^d(1-(\epsilon-1)/t)^r-(1-(\epsilon+1)(\mu/\lambda)^t)^d(1-(\epsilon+1)/t)^r \leq C/t$$

and we multiply by $(\lambda^t)^d t^r$. q.e.d.

Lemma 7.3. *Let* $\lambda > \mu > 1$ *such that* $\lambda/\mu \geq e = \exp(1)$. *Let* $d, r \in \mathbb{N}$. *There is a constant* $C > 1$ *such that, if* $t \geq 1$ *and* $0 \leq \epsilon \leq t, \lambda^t - \epsilon\mu^t \geq 0$ *then*

$$(\lambda^t)^d t^r - (\lambda^t - \epsilon\mu^t)^d (t - \epsilon)^r \leq \epsilon C (\lambda^t)^d t^{r-1}.$$

Proof. Dividing by $(\lambda^t)^d t^r$ we see that the inequality is equivalent to

$$(1 - \epsilon(\mu/\lambda)^t)^d (1 - \epsilon/t)^r \geq 1 - C\epsilon/t.$$

As $\lambda/\mu \geq e$ we have $(\mu/\lambda)^t < 1/t$ hence it is sufficient to find C such that

$$(1 - \epsilon/t)^{d+r} \geq (1 - C\epsilon/t).$$

We can choose $C = d + r$ as shown by calculus. q.e.d.

8. Solvable Lie groups

8.1 The group law in coordinates

Any simply connected solvable Lie group S is an extension

$$N \to S \to \mathbb{R}^r$$

of a simply connected nilpotent Lie group N by an an abelian Lie group $\mathbb{R}^r$ (see for example [48]). Let p be the projection. Let $X_1, ... X_r \in T_e S$ such that $T_e p(X_i) = \partial/\partial x_i$ where $\partial/\partial x_i$ for $1 \leq i \leq r$ is the canonical basis of $\mathbb{R}^r$. The projection does not split in general but we can nevertheless write any element of S in a unique way

$$x \exp(a_1 X_1) ... \exp(a_r X_r),$$

where $x \in N$ and $(a_1, ..., a_r) \in \mathbb{R}^r$. We identify N with its Lie algebra L as previously explained. We put

$$\sigma : \mathbb{R}^r \to S^r$$

$$(a_1, ..., a_r) \to (\exp(a_1 X_1), ..., \exp(a_r X_r))$$

$$a = (a_1, ..., a_r) \in \mathbb{R}^r$$

$$A = \exp(a_1 X_1)... \exp(a_r X_r) \in S.$$

We have global coordinates on S

$$L \times \mathbb{R}^r$$

$$(x, a).$$

In those coordinates the multiplication law of S is given, thanks to Lemma 6.1, by

$$(x, a)(y, b) = (xAyA^{-1}C(\sigma(a))(\sigma(b)), a + b).$$

8.2 Comparing the left-invariant metric with the Euclidean one again

We also write $p(x, a) = a$ and define $\rho(x, a) = x$. Recall that we have chosen a Euclidean structure on L. On $\mathbb{R}^r$ we choose the canonical one. On $L \times \mathbb{R}^r$ we choose the product structure. As usual, when doing computations in Euclidean spaces, for any point $(x, a) \in L \times \mathbb{R}^r$, we identify the Euclidean spaces $T_{(x,a)} L \times \mathbb{R}^r$, with $L \times \mathbb{R}^r$, the linear maps $T_{(x,a)} p$ with p, and the maps $T_{(x,a)} \rho$ with ρ.

Lemma 8.1. *There exists $\alpha > 1$ such that if $1 \leq k \leq c$, if $t \geq 1$, if $x \in L$ is such that $x = \sum_{i=1}^{c} t_i e_i(x)$ with $t_i \leq t^i$ for $i \leq k - 1$ and if $a \in \mathbb{R}^r$, then the derivative of the left translation $l_{(x,a)}$ by $(x, a) \in L \times \mathbb{R}^r$ at the identity satisfies*

$$|(\rho T_{(0,0)} l_{(x,a)}(h, u))_k| \leq \alpha \sup_{1 \leq i \leq r} \alpha^{|a_i|} t^{k-1} (|h| + |u|),$$

where $(h, u) \in T_{(0,0)} L \times \mathbb{R}^r$.

Proof. As explained above,

$$\rho l_{(x,a)}(y, b) = xAyA^{-1}C(\sigma(a))(\sigma(b)).$$

Hence taking the derivative of the composition

$$L \times \{0\} \to L \times \mathbb{R}^r \xrightarrow{l_{(x,a)}} L \times \mathbb{R}^r \xrightarrow{\rho} L$$

at the identity and remembering that $C(\sigma(a))(e,...,e) = e$, we obtain

$$\rho T_{(0,0)}l_{(x,a)}(h,0) = T_0 l_x T_0 c_A(h).$$

As L is the kernel of the projection $p : S \to \mathbb{R}^r$, the image of the map $C(\sigma(a))$, which contains only commutators, is included in L. As

$$l_x C(\sigma(a))(\sigma(b)) = \rho l_{(x,a)}(0,b),$$

taking the derivative at the identity of the composition

$$\mathbb{R}^r \xrightarrow{\sigma} S^r \xrightarrow{C(\sigma(a))} L \xrightarrow{l_x} L$$

gives

$$\rho T_{(0,0)}l_{(x,a)}(0,u) = T_0 l_x T_{(e,\dots,e)} C(\sigma(a))(u_1 X_1, ..., u_r X_r).$$

Applying Corollary 5.1 to $T_0 l_x$, Lemma 6.3 to $T_0 c_A$ and Corollary 6.1 to $T_{(e,\dots,e)}C(\sigma(a))$ completes the proof. q.e.d.

Proposition 8.1. *There exists $\nu > 0$ such that if $\lambda > \mu \geq \nu$, if $t \geq 1$, if $\epsilon \geq 0$, if $\epsilon \leq \epsilon'$ is such that $\epsilon' \leq t$, $\lambda^t - \epsilon'\mu^t \geq 0$, if (with the notation as above) $x \in L$ is such that $t_i \leq (\lambda^t - \epsilon'\mu^t)^i$ for $1 \leq i \leq c$, if $A \in S$ is such that $|a_i| \leq t - \epsilon'$ for $1 \leq i \leq r$, and if $c : [0,l] \to S$, with $0 \leq l \leq \epsilon' - \epsilon$, is a differentiable path parametrised by the Riemannian arc length with $c(0) = xA$ then*

1.

$$|pc(l)_i| \leq t - \epsilon, 1 \leq i \leq r,$$

2.

$$|\rho c(l)_i| \leq (\lambda^t - \epsilon\mu^t)^i, 1 \leq i \leq c.$$

Proof. Consider a path c parametrised by the Riemannian arc length, that is if $s \in [0,l]$ then $|\frac{d}{ds}c(s)|_{c(s)} = 1$. As the Riemannian metric is left-invariant there is

$$(h,u) \in T_{(0,0)}L \times \mathbb{R}^r$$

(depending on s) such that $|(h,u)| = 1$ and with

$$T_{(0,0)}l_{c(s)}(h,u) = \frac{d}{ds}c(s).$$

We first prove that $|pc(l)_i| \leq t - \epsilon$. According to the formula

$$(x,a)(y,b) = (xAyA^{-1}C(\sigma(a))(\sigma(b)), a+b)$$

we see that for any $(x,a) \in L \times \mathbb{R}^r$

$$pT_{(0,0)}l_{(x,a)}(h,u) = u.$$

Hence for $1 \leq i \leq r$

$$|p\frac{d}{ds}c(s)_i| = |pT_{(0,0)}l_{c(s)}(h,u)_i| = |u_i| \leq 1$$

and

$$|pc(l)_i| \leq |pc(0)_i| + \int_0^l |p\frac{d}{ds}c(s)_i|ds \leq t - \epsilon' + l \leq t - \epsilon.$$

We apply Corollary 6.1 to the simply connected group $S = L \times \mathbb{R}^r$ and the integer $r = rank(S/L)$ to get a constant $\alpha > 1$ as explained. Let $\nu = 2\alpha^2$ and let $\lambda > \mu \geq \nu$. If the class c of L is greater than 1, let $1 \leq k \leq c-1$. (If $c = 1$, let $k = 1$.) We have

$$|\rho c(l)_k| \leq |\rho c(0)_k| + \int_0^l |\rho\frac{d}{ds}c(s)_k|ds.$$

If $k = 1$, (we apply the above-proved inequalities 1) to the restrictions of the path c to $[0,s], 0 \leq s \leq \epsilon' - \epsilon$ and Lemma 8.1 to get

$$|\rho c(s)_1| \leq \lambda^t - \epsilon'\mu^t + (\epsilon' - \epsilon)\nu^t \leq \lambda^t - \epsilon\mu^t.)$$

We assume now that the inequalities 2) are true for $i \leq k$ and we show that this implies they are also true for $i \leq k+1$. Applying the induction hypothesis to the restrictions of the path c to $[0,s]$ with $0 \leq s \leq \epsilon' - \epsilon$, we obtain, if $i \leq k$,

$$|\rho c(s)_i| \leq (\lambda^t - \epsilon\mu^t)^i.$$

The inequalities 1) above and Lemma 8.1 imply that

$$|\rho\frac{d}{ds}c(s)_{k+1}| \leq \nu^t(\lambda^t - \epsilon\mu^t)^k.$$

Hence

$$|\rho c(l)_{k+1}| \leq (\lambda^t - \epsilon'\mu^t)^{k+1} + (\epsilon' - \epsilon)\nu^t(\lambda^t - \epsilon\mu^t)^k \leq (\lambda^t - \epsilon\mu^t)^{k+1}$$

according to Lemma 7.1. q.e.d.

8.3 The Lebesgue and the Haar measures

Recall that our simply connected solvable group S has global coordinates $(x, a) \in L \times \mathbb{R}^r$ in which the group law is

$$(x, a)(y, b) = (xAyA^{-1}C(\sigma(a))(\sigma(b)), a + b).$$

Recall that we have chosen a Euclidean structure on $L \times \mathbb{R}^r$. Let μ be the corresponding Lebesgue measure.

Proposition 8.2. *If S is unimodular then μ is left-invariant.*

Proof. By hypothesis, it is enough to show that μ is right-invariant. The Jacobian at the identity of the right translation by (y, b) is equal to the product

$$\det(T_0 r_y) \det(T_0 r_b),$$

where $T_0 r_y$ is the derivative at the identity of L of the right translation in L by y, and $T_0 r_b$ is the derivative at $0 \in \mathbb{R}^r$ of the translation by b. The derivative of a translation in a nilpotent group is a linear unipotent transformation, hence it has a determinant equal to 1. q.e.d.

9. Følner sequences

9.1 Neighborhoods and the co-area formula

With the same notation as in Section 5 let L be a simply connected nilpotent Lie group. We define for $t \geq 1$

$$\Omega_t = \{x \in L : x = \sum_{i=1}^{c} t_i e_i(x), t_i \leq t^i\}.$$

This is a polydisc. We normalise the Lebesgue measure on L in order to get $vol(\Omega_t) = t^d$ where d is the growth of L, that is

$$d = \sum_{k=1}^{c} k \dim L_k.$$

Let S be a simply connected solvable Lie group. We have $S = L \times \mathbb{R}^r$ for some L as above and some r. Let $\lambda > \mu$ be large enough so that Proposition 8.1 applies. We define

$$F_t = \Omega_{\lambda^t} \times I_t,$$

where

$$I_t = \{a \in \mathbb{R}^r : |a_i| \leq t, 1 \leq i \leq r\}.$$

For $t \geq 1, \epsilon \geq 0$ such that $t - \epsilon \geq 0$, $\lambda^t - \epsilon\mu^t \geq 0$, let

$$\Lambda_{t,\epsilon} = \Omega_{\lambda^t - \epsilon\mu^t} \times I_{t-\epsilon}.$$

With that notation we obtain the following.

Proposition 9.1. *There is a constant $D > 0$ such that if $t \geq 1, \epsilon \geq 0, t - \epsilon D \geq 0$, then*

$$d(\partial\Omega_t, \partial\Omega_{t-\epsilon D}) \geq \epsilon.$$

Proof. This follows from Lemma 5.2. q.e.d.

Proposition 9.2. *Let $\epsilon' \geq \epsilon \geq 0$. Then*

$$d(\partial\Lambda_{t,\epsilon'}, \partial\Lambda_{t,\epsilon}) \geq \epsilon' - \epsilon.$$

Proof. This follows from Proposition 8.1. q.e.d.

We normalise the Lebesgue measure on $\mathbb{R}^r$ such that $vol(I_t) = t^r$. Hence $vol(F_t) = (\lambda^t)^d t^r$. As an application of Proposition 8.2 the volumes induced by left-invariant Riemannian metrics are also given (up to renormalisation) by

$$vol(\Omega_t) = t^d$$

and, if S is unimodular by

$$vol(F_t) = (\lambda^t)^d t^r.$$

A left-invariant Riemannian metric on L or on S induces on $\partial\Omega_t$ or ∂F_t a structure of Riemannian manifold (with corners). We want upper bound for their volumes.

Proposition 9.3. *There exists a constant $C \geq 1$ such that if $t \geq 1$ then*

$$vol(\partial\Omega_t) \leq Ct^{d-1},$$

$$vol(\partial F_t) \leq C(\lambda^t)^d t^{r-1}.$$

Proof. We define the corona

$$C_{t,\epsilon} = \{x \in \Omega_t : d(x, \partial\Omega_t) < \epsilon\}$$

to be the inner open ϵ-neighborhood of $\partial\Omega_t$. According to Lemma 5.2 or to Proposition 9.1 there is a constant $D > 0$ such that if $\epsilon \geq 0$ and $t - \epsilon D \geq 0$ then

$$C_{t,\epsilon} \subset \Omega_t \setminus \Omega_{t-\epsilon D}.$$

Hence

$$vol(C_{t,\epsilon}) \leq t^d - (t - \epsilon D)^d.$$

This shows that there exists a constant $C \geq 1$ such that for all $t \geq 1$ and $\max(1, t/D) \geq \epsilon \geq 0$

$$vol(C_{t,\epsilon}) \leq \epsilon C t^{d-1}.$$

For each $t \geq 1$ consider the distance function to $\partial\Omega_t$

$$f : \Omega_t \to \mathbb{R}_+$$

$$f(x) = d(x, \partial\Omega_t).$$

The co-area formula shows that

$$vol(C_{t,\epsilon}) = \int_0^\epsilon vol(f^{-1}(s))ds.$$

The function

$$[0, \epsilon] \to \mathbb{R}_+$$
$$s \mapsto vol(f^{-1}(s))$$

is continuous. Hence there is a ξ between 0 and ϵ such that

$$\epsilon \times vol(f^{-1}(\xi)) = \int_0^\epsilon vol(f^{-1}(s))ds.$$

As $\partial\Omega_t$ is a compact submanifold

$$\lim_{\xi \to 0} vol(f^{-1}(\xi)) = vol(\partial\Omega_t).$$

Hence, making ϵ tend to zero, we obtain $vol(\partial\Omega_t) \leq Ct^{d-1}$. In the case of solvable groups (that is when $r > 0$) we choose $\lambda > \mu$ as in Proposition 8.1 and also such that $\lambda \geq e\mu$ so that Lemma 7.3 also applies. We define the corona

$$C_{t,\epsilon} = \{x \in F_t : d(x, \partial F_t) < \epsilon\}$$

to be the inner ϵ-neighborhood of ∂F_t. According to Proposition 9.2,

$$C_{t,\epsilon} \subset F_t \setminus \Lambda_{t,\epsilon}.$$

Hence

$$vol(C_{t,\epsilon}) \leq vol(F_t) - vol(\Lambda_{t,\epsilon})$$

so that, applying Lemma 7.3, there is a constant $C > 1$ such that

$$vol(C_{t,\epsilon}) \leq \epsilon C(\lambda^t)^d t^{r-1}.$$

For each $t \geq 1$, let $f : F_t \to \mathbb{R}_+$ defined by

$$f(x) = d(x, \partial F_t).$$

Applying the co-area formula gives

$$vol(C_{t,\epsilon}) = \int_0^\epsilon vol(f^{-1}(s))ds.$$

Making ϵ tend to zero we obtain, by the same arguments as above,

$$vol(\partial F_t) \leq C(\lambda^t)^d t^{r-1}.$$

q.e.d.

10. Discretisation

10.1 Discrete subgroups

The inequalities we need and explain here for a co-compact lattice are true in a more general setting. Namely, we could replace the Lie group with a Riemannian manifold with bounded geometry and the lattice with a suitable net. Let G be a connected Lie group with a left-invariant Riemannian metric. Let Γ be a discrete co-compact subgroup in G. Let $R > 0$ such that

$$V_R(\Gamma) = \{g \in G : d(g, \Gamma) \leq R\} = G.$$

Let $R > \delta > 0$ such that if $x, y \in \Gamma$ and $d(x, y) < 2\delta$ then $x = y$. Let $\tilde{\Gamma}$ be a small perturbation of Γ. That is $\tilde{\Gamma} = \tau(\Gamma)$ where

$$\tau : \Gamma \subset G \to G$$

is a map such that $d(\gamma, \tau(\gamma)) < \delta$ for all $\gamma \in \Gamma$. Notice that τ is injective. To a mesurable subset $\Omega \in G$ and a small perturbation $\tilde{\Gamma}$ of Γ we associate two subsets of Γ

$$Int(\omega) \subset \omega \subset \Gamma$$

defined as follows. First we consider

$$\tilde{\omega} = V_{2R}(\Omega) \cap \tilde{\Gamma},$$

$$Int(\tilde{\omega}) = \tilde{\omega} \setminus V_R(G \setminus \Omega)$$

then we define

$$\omega = \tau^{-1}(\tilde{\omega}),$$

$$Int(\omega) = \tau^{-1}(Int(\tilde{\omega})).$$

Lemma 10.1. *There is a constant $C > 1$ such that for all measurable subsets Ω,*

$$|\omega| \geq vol(\Omega)/C,$$

$$|Int(\omega)| \leq C vol(\Omega).$$

Proof. As

$$\Omega \subset \bigcup_{\gamma \in \omega} B_R(\gamma),$$

choosing $C = vol(B_R(e))$ proves the first inequality. As $\tilde{\Gamma}$ is a (uniformly) discrete subspace in G there is a constant $C > 0$ such that

$$|F| \leq C vol(\bigcup_{\tilde{\gamma} \in F} B_R(\tilde{\gamma}))$$

for any subset $F \subset \tilde{\Gamma}$. As

$$\bigcup_{\tilde{\gamma} \in Int(\tilde{\omega})} B_R(\tilde{\gamma}) \subset \Omega$$

the second inequality is proved. q.e.d.

We choose $S = S^{-1}$ a finite generating set for Γ. Recall that for $\omega \subset \Gamma$

$$\partial\omega = \{\gamma \in \omega : \exists s \in S : \gamma s \in \Gamma \setminus \omega\}.$$

Proposition 10.1. *With the above notation there exists a constant $C > 1$ such that if $\Omega \subset G$ is a submanifold of maximal dimension (with or without corners) then*

$$|\partial\omega| \leq C vol(V_1(\partial\Omega)).$$

Proof. Let $B = \max_{s \in S} d(e, s)$ where d is the Riemannian distance on G. Let $\gamma \in \partial\omega$. By definition, $d(\gamma, \Omega) \leq 2R + \delta$ and there exists $s \in S$ such that $d(\gamma s, \Omega) > 2R - \delta > 0$. As $d(\gamma s, \Omega) = d(\gamma s, \partial\Omega)$, we have

$$\begin{aligned} d(\gamma, \partial\Omega) &\leq d(\gamma, \gamma s) + d(\gamma s, \partial\Omega) \\ &\leq B + d(\gamma s, \Omega) \leq 2B + 2R + \delta. \end{aligned}$$

Hence, to each $\gamma \in \partial\omega$ we can associate a point $x \in \partial\Omega$ with $d(\gamma, x) \leq 2B + 2R + \delta$ and we may consider the ball $B_1(x) \subset V_1(\partial\Omega)$. Hence, as Γ is discrete in G it follows that there exists a constant $C > 1$ such that

$$|\partial\omega| \leq C vol(V_1(\partial\Omega)).$$

q.e.d.

10.2 Finitely generated nilpotent groups

Proposition 10.2. *Let Γ be an infinite finitely generated nilpotent group with finite generating symmetric set S. There is a constant $C > 1$ and a family $\Omega_n, n \in \mathbb{N}$ of finite subsets of Γ such that*

$$\Omega_n \subset \Omega_{n+1}, \bigcup_{n \in \mathbb{N}} \Omega_n = \Gamma,$$

$$n/C \leq |\Omega_n| \leq Cn,$$

$$n^{\frac{d-1}{d}}/C \leq |\partial\Omega_n| \leq Cn^{\frac{d-1}{d}},$$

where d is the degree of growth of Γ.

Proof. The existence of such a family is a quasi-isometric invariant among finitely generated groups [37]. Hence, taking a finite index subgroup without torsion, we can assume that Γ is a lattice in a simply connected nilpotent Lie group N [43] 2.18. Let Ω_t be the Følner sequence of N defined in Section 9. Let $R > \delta > 0$ as in Subsection 10.1. We choose a small perturbation $\tilde{\Gamma}$ of Γ as explained in Subsection 10.1 with the property that for all $t \geq 0$,

$$|\tilde{\Gamma} \cap \partial\Omega_t| \leq 1.$$

The existence of such a perturbation is clear because Γ is countable and each neighborhood of a point $x \in \partial\Omega_t$ meets uncountably many $\partial\Omega_s, s \in \mathbb{R}$. Hence the non-decreasing function

$$\mathbb{R}_+ \to \mathbb{N}$$

$$t \mapsto |\tilde{\Gamma} \cap \Omega_t|$$

has only unit jumps. Let ω_t be the subset of Γ corresponding to Ω_t and $\tilde{\Gamma}$ as explained in Subsection 10.1. According to Lemma 10.1, there is a constant $C > 1$ such that

$$|\omega_t| \geq vol(\Omega_t)/C,$$

hence

$$|\omega_t| \geq t^d/C,$$

where d is the growth degree of N. We choose $\epsilon = 3R + 1$ and apply Proposition 9.1 to get

$$d(\tilde{\omega_t}, \partial\Omega_{t+\epsilon D}) > R$$

for some $D > 0$. This shows that

$$\tilde{\omega_t} \subset \tilde{\omega}_{t+\epsilon D} \setminus V_R(N \setminus \Omega_{t+\epsilon D}) = Int(\tilde{\omega}_{t+\epsilon D}),$$

hence

$$\omega_t \subset Int(\omega_{t+\epsilon D}).$$

Applying the second inequality of Lemma 10.1 we get, for $t \geq 1$,

$$|\omega_t| \leq Ct^d$$

for some (other) constant C. Applying Proposition 10.1 and Proposition 9.1 we get

$$|\partial\omega_t| \leq Cvol(V_1(\partial\Omega_t)) \leq Cvol(\Omega_{t+D} \setminus \Omega_{t-D}) = C((t+D)^d - (t-D)^d).$$

Hence there is a constant C such that

$$|\partial\omega_t| \leq Ct^{d-1}.$$

It follows from [11] that there is a constant C such that

$$|\partial\omega_t| \geq t^{d-1}/C.$$

Now we forget the notation Ω_t for the continuous family and define the discrete family we are looking for as $\Omega_n = \omega_t$ where $t \geq 0$ is such that $|\omega_t| = n$. q.e.d.

10.3 Polycyclic groups

Proposition 10.3. *Let* Γ *be a polycyclic group of exponential growth with finite symmetric generating set* S. *There are constants* $C > 1$, $\lambda > 1$ *and* $r \in \mathbb{N}$ *and a family* $\Omega_n, n \in \mathbb{N}$ *of finite subsets of* Γ *such that*

$$\Omega_n \subset \Omega_{n+1},$$

$$\bigcup_{n \in \mathbb{N}} \Omega_n = \Gamma,$$

$$n/C \leq |\Omega_n| \leq Cn,$$

$$\frac{n}{C\log(n)} \leq |\partial\Omega| \leq \frac{Cn}{\log(n)}.$$

Proof. The proof is similar to the proof of Proposition 10.2. Up to finite index, we can assume that Γ is a discrete co-compact subgroup of a simply connected solvable Lie group S; see [43] 4.28. Let $\lambda > \mu > 1$ as in Section 9. Let $R > 0$ as in Section 10. Consider (for t large enough) the set $\Lambda_{t,\epsilon} \subset S$ defined in Section 9 with $\epsilon = 3R + 1$. Let $\omega_{t,\epsilon} \subset \Gamma$ be the set associated with $\Lambda_{t,\epsilon}$ and a small perturbation $\tilde{\Gamma}$ of Γ as in the proof of Proposition 10.2. By Lemma 10.1 there is a constant $C > 1$ such that $|\omega_{t,\epsilon}| \geq vol(\Lambda_{t,\epsilon})/C$. Hence $|\omega_{t,\epsilon}| \geq (\lambda^t)^d t^r / C$ where $C > 1$ is (another) constant not depending on t (being large enough). According to Proposition 9.2,

$$d(\partial\Lambda_{t,\epsilon}, \partial\Lambda_{t,0}) \geq \epsilon > 3R.$$

Hence $\omega_{t,\epsilon} \subset Int(\omega_{t,0})$. Applying the second inequality of Lemma 10.1 we get $|\omega_{t,\epsilon}| \leq C(\lambda^t)^d t^r$ for some constant C. Applying Proposition 10.1, Proposition 9.2 and Lemma 7.2 we get

$$|\partial\omega_{t,\epsilon}| \leq C(\lambda^t)^d t^{r-1}$$

for some constant C. According to [11], as the growth of Γ is exponential, there is a constant C such that

$$|\partial\omega_{t,\epsilon}| \geq (\lambda^t)^d t^{r-1}/C.$$

For n large enough, let t be such that $|\omega_{t,\epsilon}| = n$. We define

$$\Omega_n = \omega_{t,\epsilon}.$$

q.e.d.

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